Physics 618 2020

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Continue with the symmetries of charged particle on a ring around a solenoid.

$$S = \int \frac{1}{2} I \dot{\phi}^2 \, dt + \frac{eB}{2\pi} \dot{\phi} \, dt$$

$$\phi \sim \phi + 2\pi$$

$$\Phi(t) = e^{i\phi(t)}$$

$$H_{\mathcal{B}} = \frac{\hbar^2}{2I} \left(-i \frac{\partial}{\partial \phi} - \mathcal{B} \right)^2$$

acting on $L^2(S^1)$

Classical system has $O(2)$ symmetry

$$R(\alpha) \, , \quad P^2 = P$$

$$P R(\alpha) P = R(\alpha)^{-1} = R(-\alpha)$$

Complete set of eigenvectors of H_B

$$\Psi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \qquad \frac{\phi \to \phi + \alpha}{e^{im\alpha}}$$

$$E_m = \frac{\hbar^2}{2I}(m - B)^2 \qquad m \in \mathbb{Z}.$$

Realize the symmetry op's quantum-mechanically

$$\boxed{\begin{aligned} R(\alpha)\cdot\psi_m &= e^{im\alpha}\psi_m \\ P\cdot\psi_m &= \psi_{2B-m} \end{aligned}}$$

$\boxed{\text{only if } 2B \in \mathbb{Z}}$

Naively $\phi \to -\phi$ $\quad \psi_m \to \psi_{-m}$

but these have different ev's

But now:

$$R(\alpha) R(\beta) = R(\alpha+\beta)$$

$$P^2 = 1$$

$$\circledast \quad P R(\alpha) P = e^{i 2\mathcal{B}\alpha} R(-\alpha).$$

$\mathcal{G}_{\mathcal{B}}$ = group of operators generated by $R(\alpha)$ $\alpha \sim \alpha + 2\pi$, P, and $z \cdot \mathbb{1}_{\mathcal{H}}$ $z \in U(1)$

$$1 \to U(1) \to \mathcal{G}_{\mathcal{B}} \longrightarrow O(2) \longrightarrow 1$$

$\boxed{2\mathcal{B} \in \mathbb{Z}}$ $\to$ $2\mathcal{B} \in 2\mathbb{Z}$ $(\mathcal{B} \in \mathbb{Z})$

$\to$ $2\mathcal{B} \in 2\mathbb{Z}+1$ $(\mathcal{B} \in \mathbb{Z} + 1/2)$

$\mathcal{B} \in \mathbb{Z}$:

$$\boxed{\tilde{R}(\alpha) = \underbrace{e^{-i\mathcal{B}\alpha}}_{} R(\alpha)}$$

makes sense

$\alpha \sim \alpha + 2\pi$

$* \Longleftrightarrow$

$$P\,\tilde{R}(\alpha)\,P = \tilde{R}(\alpha)^{-1}$$

So rotation in target space, $\phi \longrightarrow \phi + \alpha$ is implemented by $\tilde{R}(\alpha)$.

But when $2B$ odd

Introduce a group Spin(2)

As an abstract group

$$\mathrm{Spin}(2) \cong U(1) \cong SO(2) \cong \mathbb{R}/\mathbb{Z}$$

Spin(2) is different from SO(2) because it comes with a double covering:

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(2) \longrightarrow SO(2) \to 1$$

Define $\mathrm{Spin}(2) = \left\{ \exp(\hat{\alpha}\sigma^1\sigma^2) \;\middle|\; \hat{\alpha} \sim \hat{\alpha} + 2\pi \right\}$

$\hat{R}(\hat{\alpha}) :=$

$$\exp(\hat{\alpha}\sigma^1\sigma^2) = \exp(i\hat{\alpha}\sigma^3)$$

$$= \cos\hat{\alpha} + i\sin\hat{\alpha}\,\sigma^3$$

Recall $SU(2) \xrightarrow{\pi} SO(3)$

$$u\,\vec{x}\cdot\vec{\sigma}\,u^{-1} = (R(u)\cdot\vec{x})\cdot\vec{\sigma}$$

Restrict to $u = \cos\hat{\alpha} + i\sin\hat{\alpha}\,\sigma^3$

$$\vec{x}\cdot\vec{\sigma} = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$$

Check $\pi(\exp\hat{\alpha}\sigma^1\sigma^2)$ is rotation by $2\hat{\alpha}$ around z-axis.

$$\pi(\hat{R}(\hat{\alpha})) = R(2\hat{\alpha})$$

$$\mathrm{Pin}^+(2) := \mathrm{Spin}(2) \rtimes \mathbb{Z}_2$$

$$\sigma = \hat{P}$$

$$\sigma \hat{R}(\hat{\alpha}) \sigma = \hat{R}(\hat{\alpha})^{-1} = \hat{R}(-\hat{\alpha}).$$

If $2B$ is odd we have an isomorphism

$$\cancel{\mathcal{G}_B \cong U(1) \times \mathrm{Pin}^+(2)}$$

$$\begin{tikzcd}
1 \arrow[r] & \{\pm 1\} \arrow[r] \arrow[d, hook] & \mathrm{Pin}^+(2) \arrow[r] \arrow[d, hook] & O(2) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & U(1) \arrow[r] & \mathcal{G}_B \arrow[r] & O(2) \arrow[r] & 1
\end{tikzcd}$$

odd implementing rotation by $2\hat{\alpha}$

$$\rho \quad \begin{array}{ccc} z & \longrightarrow & z \cdot \mathbb{1}_{\mathcal{H}} \\ \hat{R}(\hat{\alpha}) & \longrightarrow & e^{-i(2B)\hat{\alpha}} R(2\hat{\alpha}) \end{array}$$

because Spin(2) double covers SO(2)

$$\hat{P} \longrightarrow \mathcal{P}$$

$$\rho\left(\hat{P}\,\hat{R}(\hat{\alpha})\,\hat{P}\right) = \rho\left(\hat{R}(\hat{\alpha})^{-1}\right)$$

gives a true relation thanks to

$$\mathcal{P}\,R(\alpha)\,\mathcal{P} = e^{i\,2B\alpha}\,R(-\alpha).$$

Summary

1. Classical theory has $O(2)$ symmetry
2. $2B \notin \mathbb{Z}$ Quantum Theory only has $SO(2)$. No quantum analog of P. See that from spectrum of H_B: all eigenspaces are 1-dim.
3. <u>$2B$ even</u>: Sequence splits and $\mathcal{G}_B \cong U(1) \times O(2)$

   $O(2)$ remains a quantum symmetry realized by $\mathcal{P}$ and $\tilde{R}(\alpha)$
4. <u>$2B$ odd</u>: sequence does not split

$$
\begin{array}{ccccccccc}
1 & \rightarrow & \mathbb{Z}_2 & \rightarrow & Pin^+(2) & \rightarrow & O(2) & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
1 & \rightarrow & U(1) & \rightarrow & \mathcal{G}_B & \rightarrow & O(2) & \rightarrow & 1
\end{array}
$$

$$\rho(\hat{R}(\hat{\alpha})) = e^{-i2\mathcal{B}\hat{\alpha}} \mathcal{R}(2\hat{\alpha})$$

$$\rho(\hat{P}) = \mathcal{P}$$

Rep. of $Pin^+(2)$ on $\mathcal{H}$, commuting with $H_{\mathcal{B}}$.

Remarks

1. Particle we put on the ring did not have any intrinsic spin.

$$H = \frac{\mathcal{L}^2}{2I} \qquad \mathcal{L} = \text{angular momentum}$$

$2\mathcal{B}$ odd quantum system has $\frac{1}{2}$-integral spin!

Example of topological terms inducing fractional quantum numbers.

2. Spin and Pin Groups

Clifford algebra is an algebra

(Vector space V w/ multiplication of vectors $V \times V \to V$)

$e_1, \dots, e_d$ basis for V

Suppose $\exists$ quadratic form

$Q : V \times V \to k$ ← field

$Q(e_i, e_j) = Q_{ij}$ nondeg. = invertible.

Relations

$$\boxed{e_i e_j + e_j e_i = 2 Q_{ij} \quad *}$$

Now you "multiply" vectors

$$\sum_{k, i_1 \dots i_k} c_{i_1 \dots i_k} e_{i_1} e_{i_2} \cdots e_{i_k} \in Cliff(Q)$$

provided we use relations $*$

$$\mathrm{Cliff}(Q) = \frac{T^\bullet V}{\left(e_i e_j + e_j e_i - 2Q_{ij} \cdot \mathbb{1} \right)}$$

$$\longrightarrow \{\gamma_i, \gamma_j\} = 2Q_{ij} \cdot \mathbb{1}.$$

For a vector $v = v^i e_i$ $\quad$ e_i basis for V

define $\gamma(v) := v^i \gamma_i \longleftarrow$

$v^i e_i \in \mathrm{Cliff}(Q).$

exercise

$$(\gamma(v))^2 = Q(v) \cdot \mathbb{1}.$$

If $Q_{ij} = \delta_{ij}$ $\quad 1 \leq i, j \leq d$

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

$$\underline{\underline{\mathrm{Pin}^+(d)}} := \left\{ \pm \underline{\gamma(v_1) \cdots \gamma(v_r)} \;\middle|\; \underline{\underline{v_i^2 = 1}} \right\}$$

$$1 \to \mathbb{Z}_2 \to \mathrm{Pin}^+(d) \xrightarrow{\pi} O(d) \to 1$$

generalize our covering map

$$u\, \vec{x}\cdot\vec{\sigma}\, u^{-1} = (\pi(u)\vec{x})\cdot\vec{\sigma}$$

$$(-1)^r g\, \gamma(w)\, g^{-1} = \gamma(\pi(g)\cdot w)$$

$w \in V$, $g \in \mathrm{Pin}^+(d)$

$$= \{\pm \gamma(v_1)\cdots\gamma(v_r) \mid v_i^2 = 1,\ r \in \mathbb{Z}_+\}$$

$g = \gamma(v)$ for some v, $v^2 = 1$

$\in \mathrm{Pin}^+(d)$, $r = 1$

$$-\gamma(v)\gamma(w)\gamma(v)^{-1} = \begin{cases} \gamma(-w) & \text{if } w = \alpha v \\ \gamma(w) & \text{if } w \perp v \end{cases}$$

$$= \gamma(R_v(w))$$

$\mathrm{Spin}(d) \subset \mathrm{Pin}^+(d)$

$$\left\{ \pm \gamma(v_1) \cdots \gamma(v_{2r}) \;\middle|\; \begin{array}{l} v_i^2 = 1 \\ r \in \mathbb{Z}_+ \end{array} \right\}$$

Projects under π to a product of <u>even</u> # of reflections

$$1 \to \{\pm 1\} \to \mathrm{Spin}(d) \xrightarrow{\pi} SO(d) \to 1$$

Irrep of $C\ell(3)$ $\quad \gamma_1 = \sigma^1 \quad \gamma_2 = \sigma^2 \quad \gamma_3 = \sigma^3$

$$\mathrm{Spin}(3) \cong SU(2)$$

but for $d > 3$

$\mathrm{Spin}(d)$ is not an orthogonal or unitary group!

$Pin^-(d)$? $\quad Q_{ij} = -\delta_{ij}$

$$\{\gamma_i, \gamma_j\} = -2\delta_{ij}$$

$k = \mathbb{R}$ nonisomorphic Clifford algebra

$Cl(-d) \not\cong Cl(d)$ (in general).

$$Pin^-(d) = \left\{ \pm \gamma(v_1) \cdots \gamma(v_r) \;\middle|\; \begin{array}{l} v_i^2 = -1 \\ r \in \mathbb{Z}_+ \end{array} \right\}$$

Our $Pin^+(2)$ is a special case of the above:

<u>Spin(2)</u>: $\quad \underbrace{(v^1\sigma^1 + v^2\sigma^2)(w^1\sigma^1 + w^2\sigma^2)}_{\vec{v}\cdot\vec{w} + (\vec{v}\wedge\vec{w})\sigma^1\sigma^2}$

$\vec{v}^{\,2} = 1 \quad \vec{w}^{\,2} = 1$

$$\vec{v}\cdot\vec{w} + (\vec{v}\wedge\vec{w})\sigma^1\sigma^2 = \cos\hat{\alpha} + \sin\hat{\alpha}\,\sigma^1\sigma^2 = \hat{R}(\hat{\alpha})$$

Check:

$Pin^+(2) \cong Spin(2) \rtimes \mathbb{Z}_2$

The difference between $Pin^+(d)$ and $Pin^-(d)$:

In both cases the reflection of w in the hyperplane $\perp v$ is realized by :

$$-\gamma(v)\,\gamma(w)\,\gamma(v)^{-1} = \gamma(R_v(w))$$

$$1 \to \{\pm 1\} \to Pin^+(d) \xrightarrow{\pi^+} O(d) \to 1$$

$$1 \to \{\pm 1\} \to Pin^-(d) \xrightarrow{\pi^-} O(d) \to 1$$

the lift of reflection $R_v \in O(d)$ is

$$(\pi^\pm)^{-1} R_v = \{\pm\gamma(v)\}$$

In Pin^+ $\quad \gamma(v)^2 = +1$

In Pin^- $\quad \gamma(v)^2 = -1$

3. <u>2B odd we have $Pin^+(2)$ symmetry, not $O(2)$ symmetry.</u>

General fact about symmetry in QM:

We have a projective rep

$$\rho : G \longrightarrow GL(\mathcal{H})$$

just a true rep. of some other group $\widetilde{G} = U(1)$ c.e. of G

WLOG

$$\boxed{\rho : G \longrightarrow GL(\mathcal{H}) \quad \text{true rep}}$$

<u>Symmetry</u>: One time evolution to another.

If we have dynamics, e.g. a Hamiltonian

$$U(t) = e^{-it/\hbar H}$$

$$\boxed{U(g)\,U(t) = U(t)\,U(g)}$$

(as long as g doesn't change the orientation of time)

$$[U(g), H] = 0.$$

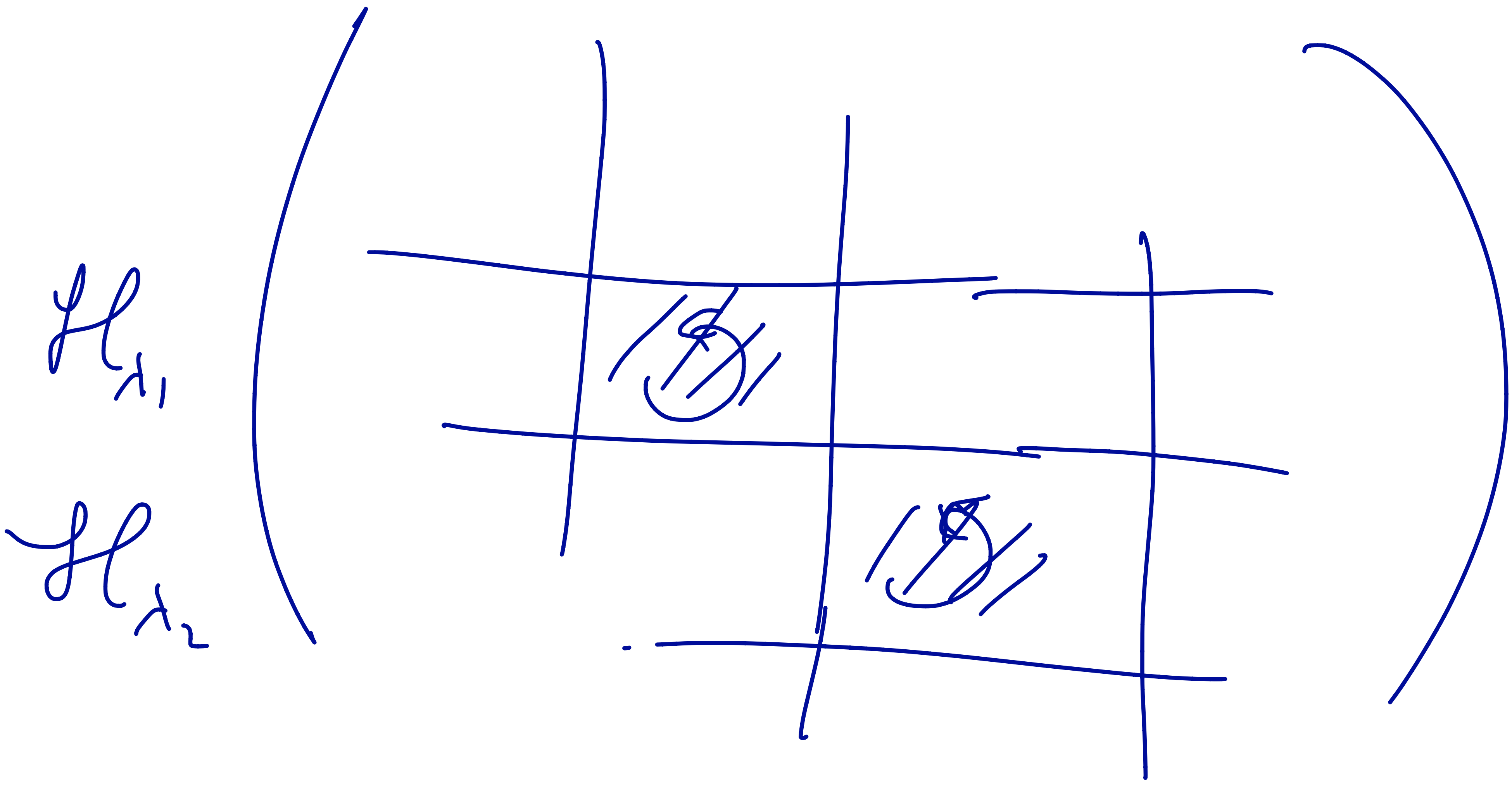
If $\mathcal{H}_\lambda$ is an eigenspace of H: $\quad H\psi = E_\lambda \psi, \quad \psi \in \mathcal{H}_\lambda$

Then $\mathcal{H}_\lambda \subset \mathcal{H}$ is a rep. space of G.

$$H\,U(g)\psi = U(g)\,H\psi = E_\lambda\, U(g)\psi$$

So $U(g) : \mathcal{H}_\lambda \longrightarrow \mathcal{H}_\lambda$

- Symmetry helps block diagonalize Hamiltonians



$B = 1/2$ ground state is a Q-bit

$$E_m = \frac{\hbar^2}{2I}(m - B)^2$$

ground state is spanned by ψ_0 and ψ_1 2-dim'l

$$\mathrm{Span}\{\psi_0, \psi_1\} = \mathcal{H}_\lambda$$

$$\lambda = \frac{\hbar^2}{8I}$$

must be a rep. of $Pin^+(2)$.

$$\rho(\hat{R}(\hat{\alpha})) = e^{-i\,2\theta\alpha}\, R(2\hat{\alpha})$$

you check relative to ordered basis $\{\psi_0, \psi_1\}$ matrix rep

$$\boxed{\begin{aligned} \rho(\hat{R}(\hat{\alpha})) &= \begin{pmatrix} e^{-i\hat{\alpha}} & 0 \\ 0 & e^{i\hat{\alpha}} \end{pmatrix} \\ \rho(\hat{P}) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}}$$

well defined

Note $\hat{\alpha}$ it induces rotation by $\alpha = 2\hat{\alpha}$

So a rotation by α would be represented by

$$\begin{pmatrix} e^{-i\alpha/2} & \\ & e^{i\alpha/2} \end{pmatrix}$$

Not well-defined because $\alpha \sim \alpha + 2\pi$. $\mathcal{H}_{1,\text{gnd}}$ is <u>not</u> a rep of $O(2)$ but it <u>is</u> a rep of $\mathrm{Pin}^+(2)$.

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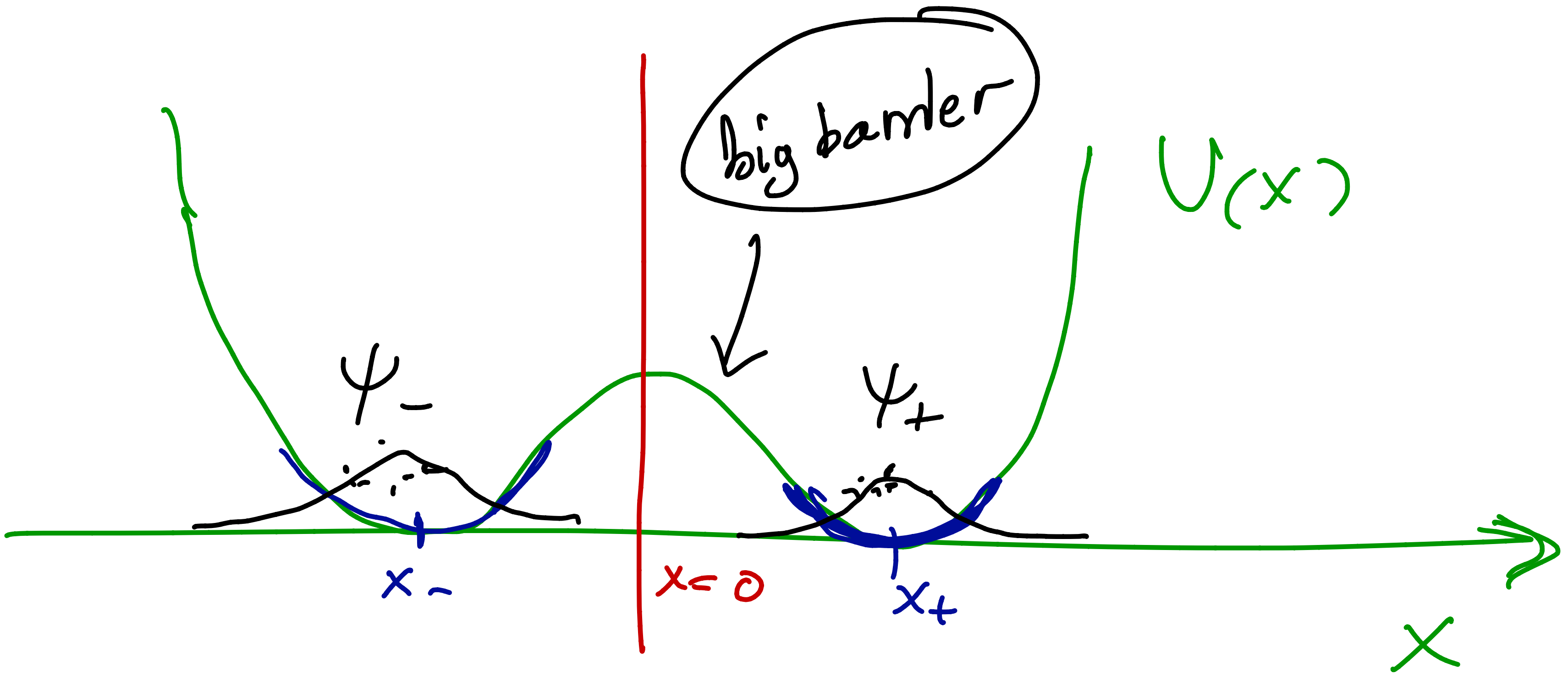
4 preface:

Consider QM of a particle on the line in double well potential



$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + U(x)$$

$$\langle \psi_- | H | \psi_+ \rangle \neq 0$$

acts on $L^2(\mathbb{R}) = \mathcal{H}$

$U(x) = U(-x)$ $\quad \mathbb{Z}_2$ symmetry

Realized on $\mathcal{H}$ $\quad P: \psi(x) \longrightarrow \psi(-x)$

$$[P, H] = 0 \qquad \mathcal{H} = \underset{P=+1}{\mathcal{H}^+} \oplus \underset{P=-1}{\mathcal{H}^-}$$

In perturbation theory the groundstate is 2-fold degenerate

Instanton / Tunneling effects modify the approximate expressions and the true Q.M. groundstate is ONE DIMENSIONAL (and in $\mathcal{H}^+$)

Now lets consider our particle on a ring

put a potential $U(\phi)$

$$U(\phi) = \sum_{n \in \mathbb{Z}} C_n \cos(n\phi) + S_n \sin(n\phi)$$

Completely breaks classical $O(2)$ symm.

But we can restrict to a special set of potentials (still ctbly many)

$$U(\phi) = \sum_n u_n \cos(2n\phi)$$

Classical $O(2)$ symmetry is broken to $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by

$p: \phi \to -\phi$ (only cosines)

$r: \phi \to \phi + \pi$ $R(\pi)$

$$p^2 = 1, \quad r^2 = 1, \quad pr = rp$$

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[d] & ?\,D_4 \arrow[r] & \mathbb{Z}_2 \times \mathbb{Z}_2 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{Z}_2 \arrow[r] & \mathrm{Pin}^+(2) \arrow[r] & O(2) \arrow[r] & 1
\end{tikzcd}$$

$\{\hat{R}(\pi/2), P\} = D_4$

$R(\pi)$

Restrict this extension to

$\mathbb{Z}_2 \times \mathbb{Z}_2 \subset O(2)$ $\langle R(\pi), P \rangle$

If we believe the cocycle is continuous as a function of u_n

(We know cocycle at $u_n = 0 \ \forall n$)

For 2B odd we will get the extension

Explain next time

$$1 \to \mathbb{Z}_2 \to D_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1$$

Prediction: even for $u_n \neq 0$

There will be operators

$\mathcal{R}, \mathcal{P}$ commuting with

$H_{\mathcal{B}, u_n}$ satisfying D_4 relations

$$\mathcal{R}^4 = 1, \ \mathcal{P}^2 = 1, \ \mathcal{P}\mathcal{R}\mathcal{P} = \mathcal{R}^{-1}$$